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Hahn echo and criticality in spin-chain systems

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Abstract. We present a theory to establish a relation between Hahn spin-echo of a spin-1/2 particle and quantum phase transitions in many-body systems. The Hahn echo is calculated and discussed at zero as well as at finite temperatures. On the example of XY model, we show that the critical points of the chain are marked by the extremal values in the Hahn echo, and can influence the Hahn echo in finite temperatures. An explanation for the relation between the echo and criticality is also presented.

PACS. 03.65.Ud Entanglement and quantum nonlocality – 05.70.Jk Critical point phenomena

1 Introduction

Quantum phase transitions [1] (QPTs) have attracted enormous attention within various fields of physics in the past decade. They exist on all length scales, from microscopic to macroscopic. Because QPTs, which describe transitions between quantitatively distinct phases, are driven solely by quantum fluctuations, they provide valuable information about the ground state and nearby excited states of quantum many-body systems. The observation of quantum criticality depends eventually on the experimentally available temperature, then it is natural to ask how high in temperature can the effects of quantum criticality persist? Do quantum critical points shed light on quantum mechanics of macroscopic systems, for instance providing a deeper understanding of decoherence? In this paper we answer these questions by analyzing the density matrix of a spin-1/2 particle coupled to a many-body system. The results suggest that the Hahn echo of the spin-1/2 particle can be used as a marker for the criticality in the many-body system in the weak spin-system coupling limit. As an example, we calculate the Hahn echo of a spin-1/2 particle, taken the XY spin-chain as the many-body system.

The Hahn echo was first introduced by Hahn [2] to observe and measure directly transverse relaxation time T_2, i.e., the dephasing time. It differentiates from the Loschmidt echo in that the latter measures the sensitivity of quantum system dynamics to perturbations in the Hamiltonian. For a certain regime of parameters, the Loschmidt echo decays exponentially with a rate given by the Lyapunov exponent of the underlying classically chaotic system. Recently, a huge interest was attracted in the attempt of characterizing QPTs in terms of entanglement, by analyzing extremal points, scaling and asymptotic behavior in various entanglement measures [3–7]. The relation between Berry's phases and quantum critical points was also established recently in the XY model [8–10]. In this paper, we shall show how critical points can be reflected in the Hahn spin-echo of a spin-1/2 particle that couples to the many-body system, and what is the finite temperature effect on the Hahn spin-echo.

The structure of this paper is organized as follows. In Section 2 we present a theory to calculate and analyze the reduced density matrix of the spin-1/2 particle coupled to a quantum many-body system, and establish a relation between the density matrix and criticality. Two cases of couplings are discussed in the analysis. An example, which describes a spin-1/2 particle coupled to the XY spin chain, to detail the representation is given in Section 3. And finally in Section 4, we present a brief remark on the low temperature effect on the Hahn echo and conclude our results.

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2 Formulation

In this section, we will give a formalism to analyze the reduced density matrix of a spin-1/2 particle coupled to a quantum many-body system, and establish a relation between the reduced density matrix of the spin and the criticality in the quantum system. Three cases of coupling are considered. We first consider the case where the coupling conserves the energy of the spin-1/2 particle. Then we analyze the case when the energy of the spin-1/2 particle does not conserve, but the energy of the quantum many-body system does. Finally, we perturbatively analyze a general case where the couplings do not conserve energies of neither the spin nor the many-body system. The first case correspond to dephasing in the spin-1/2 particle, while the second and third kinds of coupling result in dissipation in the particle.

Consider the spin-1/2 particle coupled to a quantum many-body system. The Hamiltonian that governs the evolution of the whole system may have the form

$$H = H_s + H_c + H_i, \tag{1}$$

where $H_s = \frac{\mu}{2}\sigma^z$, describes the free Hamiltonian of the spin-1/2 particle, $H_c = H_0 + \lambda H_1$ stands for the free Hamiltonian of the quantum system, $H_i = gH_2 \otimes \sigma^z$ represents the couplings between them, and H_2 denotes an arbitrary operator of the quantum system. It is clear that $[H_i, H_s] = 0$, therefore the energy of the spin-1/2 particle conserves. The quantum system described by H_c undergoes a quantum phase transition for parameter $\lambda = \lambda_c$. It is easy to show that the time evolution operator for the whole system may be written as

$$U(t) = U_\uparrow(t)|\uparrow\rangle\langle\uparrow| + U_\downarrow|\downarrow\rangle\langle\downarrow|, \tag{2}$$

with $U_\uparrow(t)$ and $U_\downarrow(t)$ satisfying

$$i\hbar\frac{\partial}{\partial t}U_{\uparrow,\downarrow}(t) = H_{\uparrow,\downarrow}U_{\uparrow,\downarrow}(t), \tag{3}$$

where

$$H_{\uparrow/\downarrow} = H_c \pm \left(\frac{\mu}{2} + gH_2\right). \tag{4}$$

$|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of σ^z. Having these expressions, we now show that the off-diagonal elements of the reduced density matrix of the spin-1/2 particle change dramatically in the vicinity of critical points. To this end, we assume that the spin-1/2 particle and the quantum many-body system are initially independent, such that we take the following product state as the initial state

$$|\psi(0)\rangle = |\psi_s(0)\rangle \otimes |G_c\rangle, \tag{5}$$

where $|G_c\rangle\rangle$ represent the ground state of H_c and $|\psi_s(0)\rangle = s_\uparrow|\uparrow\rangle + s_\downarrow|\downarrow\rangle$ with $|s_\downarrow|^2 + |s_\uparrow|^2 = 1$. By a standard calculation, we give the reduced density matrix of the spin-1/2 particle as follows

$$\rho_s = \begin{pmatrix} |s_\uparrow|^2 & s_\uparrow s_\downarrow^* \Gamma(t) \\ s_\uparrow^* s_\downarrow \Gamma^*(t) & |s_\downarrow|^2 \end{pmatrix}. \tag{6}$$

Here, $\Gamma(t)$ is defined by

$$\begin{aligned}\Gamma(t) &= \langle G_c|U_\uparrow(t)U_\downarrow^\dagger(t)|G_c\rangle \\ &= \langle G_c|e^{-iH_\uparrow t}e^{iH_\downarrow t}|G_c\rangle,\end{aligned} \tag{7}$$

this expression is the survival probability of the ground state of the quantum system under the action of the Hamiltonian $H_\uparrow$ and $H_\downarrow$. The leading term $\Gamma(t) \simeq \langle G_c^\uparrow|G_c^\downarrow\rangle$ of this equation represents the overlap function between two ground states $|G_c^\uparrow\rangle$ and $|G_c^\downarrow\rangle$ corresponding to two different Hamiltonian $H_\uparrow$ and $H_\downarrow$, respectively. This overlap function was shown [16] to take extremal values in the vicinity of critical points. Thus the Hahn echo which characterizes the transverse relaxation time would behave dramatically at the critical points. In fact, $|\Gamma(t)|$ as a function of T_2 characterizes the dephasing of the spin-1/2 particle. Thus the critical points will be reflected in the Hahn echo that is a function of the elements of the density matrix. In fact, $\Gamma(t)$ also represents the Loschmidt echo which goes exponentially to zero with the parameters approaching the critical points [13], where the bath was modeled as the Ising spin-chain and the spin to bath coupling is of pure dephasing, while in reference [14] the bath was described by the one-dimensional XY spin-chain. Going beyond the case of a central spin coupled uniformly to all the spins in the bath, Rossini et al. have studied decoherence induced by the spin-chain in a two-level system, a relation between the decoherence and entanglement inside the chain was also established. We would like to note that $[H_c, H_2] = 0$ leading to $[H_\uparrow, H_\downarrow] = 0$. This kind of spin-system coupling results in $|\Gamma(t)| = 1$, thus the Hahn echo can not signal the critical points of the quantum system.

Next we turn to study the case when the energy of the spin-1/2 particle does not conserve, but the energy of the many-body system does. This implies that $[H_s, H_i] \neq 0$, and $[H_i, H_c] = 0$. Without loss of generality, we consider the following Hamiltonian

$$\begin{aligned}H &= H_s + H_c(\lambda) + H_i, \\ H_s &= \Delta\sigma^z, \\ H_i &= (g_x\sigma^x + g_y\sigma^y + g_z\sigma^z) \otimes H_c(\lambda).\end{aligned} \tag{8}$$

The fact that the interaction Hamiltonian H_i commutes with H_c enables us to write the time evolution operator as

$$U(t) = \sum_{n=1}^{N} U_n(t)|E_n(\lambda)\rangle\langle E_n(\lambda)|, \tag{9}$$

where $\{|E_n(\lambda)\rangle\}$ are eigenstates of H_c with corresponding eigenenergies $\{E_n = E_n(\lambda)\}$, and were assumed to be nondegenerate. It is easy to show that

$$\begin{aligned}U_n(t) &= e^{-iE_nt}e^{-iH_nt}, \\ H_n &= \sqrt{g_x^2E_n^2 + g_y^2E_n^2 + (\Delta + g_zE_n)^2} \\ &\quad \times e^{i\sigma^z\gamma_n}e^{i\sigma^x\beta_n}\sigma^z e^{-i\sigma^x\beta_n}e^{-i\sigma^z\gamma_n}.\end{aligned} \tag{10}$$

Here,

$$\cos(2\gamma_n) = \frac{g_y}{\sqrt{g_y^2 + g_x^2}},$$

and

$$\cos(2\beta_n) = \frac{\Delta + g_zE_n}{\sqrt{E_n^2(g_x^2 + g_y^2) + (\Delta + g_zE_n)^2}}.$$

Quantum phase transition theory tells us that the ground state energy $E_0(\lambda)$ behaves dramatically in the vicinity of the critical point λ_c, this would reflect in $U_0(t)$ that is a function of $E_0(\lambda)$. Having established this linkage, we claim that the final state of the spin-1/2 particle

$$\rho_s(t) = \sum_n |c_n|^2 U_n(t)\rho_s(0)U_n^\dagger(t), \tag{11}$$

and the Hahn echo can signal quantum critical points in the many-body system, provided the initial state of the many-body system is a superposition of $\{|E_n\rangle\}$, i.e., $|\psi_c(0)\rangle = \sum_{n=0}^{N} c_n|E_n\rangle$ ($c_0 \neq 0$), and at least two c_n including c_0 are not zero. We would like to notice that the Hahn echo in the second case can not signal critical points, if the many-body system is initially in the ground state with probability 1. This is because the spin-system interaction could not excite the many-body system in this case. In other words, if the couplings between the spin-1/2 and the many-body system commute with the free Hamiltonian of the many-body system, the many-body system would remain in its initial state if the initial state is one of the eigenvalues of the free Hamiltonian H_c. Thus, the many-body system would make no effect on the spin-1/2 particle during the dynamics, and consequently the Hahn-echo could not signal the critical points of the many-body system. This discussion holds for all cases with H_i satisfying $[H_c, H_i] = 0$.

The above discussions can be straightforwardly extended to a wide range of spin to many-body interactions, as follows. Consider a general spin to many-body system coupling H_i in equation (1), $[H_i, H_c] \neq 0$, and $[H_i, H_s] \neq 0$. The time evolution operator $U(t)$ in this case could not be written in neither equation (2) nor equation (9). But it may take the form,

$$U(t) = \sum_{\alpha,\beta} U_{\alpha\beta}(t)|\alpha\rangle\langle\beta|, \tag{12}$$

with $\alpha, \beta =\uparrow, \downarrow$. This is a general case without restriction on H_i. It is easy to show that $U_{\alpha\beta}(t)$ satisfy

$$i\hbar\frac{\partial}{\partial t}U_{\alpha\beta}(t) = \left(H_c + \frac{\mu_\alpha}{2} + \langle\alpha H_i|\alpha\rangle\right)U_{\alpha\beta}(t) + \langle\bar{\alpha}|H_i|\alpha\rangle U_{\bar{\alpha}\beta}(t), \tag{13}$$

where $H_s|\alpha\rangle = \frac{\mu_\alpha}{2}|\alpha\rangle$, $\bar{\alpha} =\uparrow, \downarrow$ and $\bar{\alpha} \neq \alpha$. In order to keep the critical properties of the many-body system unchanged, small coupling H_i is required. This leads to

$$U_{\alpha\beta}(t) \simeq U^{(0)}_{\alpha\beta}(t) = e^{-\frac{i}{\hbar}(H_c + \frac{\mu_\alpha}{2} + \langle\alpha|H_i|\alpha\rangle)t}. \tag{14}$$

To get this result, terms with $\langle\bar{\alpha}|H_i|\alpha\rangle$ had been ignored. Up to first order in $\langle\bar{\alpha}|H_i|\alpha\rangle$, $U_{\alpha\beta}(t)$ is

$$U^{(1)}_{\alpha\beta}(t) \simeq U^{(0)}_{\alpha\beta}(t)(1 + \hat{\delta}), \tag{15}$$

where $\hat{\delta} = \frac{\langle\bar{\alpha}|H_i|\alpha\rangle(\langle\bar{\alpha}|H_i|\alpha\rangle)^\dagger}{\langle\alpha|H_i|\alpha\rangle - \langle\bar{\alpha}|H_i|\bar{\alpha}\rangle + 0.5(\mu_\alpha - \mu_{\bar{\alpha}})}$. We note that $\langle\alpha|H_i|\alpha\rangle$ is an operator for the many-body system. Therefore, the ground state of the many-body system will be perturbed by $\langle\alpha|H_i|\alpha\rangle$ and $\hat{\delta}$, resulting in sharp changes in the elements of the reduced density matrix at the critical points. This can be shown by calculating the elements of the reduced density matrix ρ_s as,

$$\rho_s^{\alpha\beta}(t) = \langle\alpha|\rho_s(t)|\beta\rangle = \sum_{\mu,\nu=\uparrow,\downarrow}\langle G_c|U_{\alpha\mu}(t)U^\dagger_{\beta\nu}(t)|G_c\rangle s_\mu s^*_\nu, \tag{16}$$

with the same initial state as in equation (5). As we analyzed before, $\langle G_c|U_{\alpha\mu}(t)U^\dagger_{\beta\nu}(t)|G_c\rangle$ represents the overlap function between two ground states with two different values of parameters. This overlapping function behaves dramatically at the critical points, leading to sharp changes in Hahn echo in the vicinity of critical points.

3 Example

In this section, we will present two examples to detail the general formalism. The first example represents the spin-1/2 particle dephasingly coupling to the quantum many-body system, while the second one shows the Hahn echo of the spin-1/2 particle dissipatively coupling to the quantum system.

We start with the first example, where a spin-chain described by the one-dimensional XY model is taken as the quantum many-body system, the Hamiltonian for the total system (spin-1/2 particle+chain with dephasing interactions) may be given by

$$H = H_s + H_c + H_i, \tag{17}$$

where

$$\begin{aligned} H_s &= \mu s^z, \\ H_c &= -2\sum_{l=1}^{N}((1+\gamma)s_l^x s_{l+1}^x + (1-\gamma)s_l^y s_{l+1}^y + \lambda s_l^z), \\ H_i &= 4g\sum_{l=1}^{N} s^z s_l^z. \end{aligned} \tag{18}$$

Here s denotes spin operator of the system particle which couples to the chain spins s_l ($l = 1, ..., N$) located at the lattice site l. The spins in the chain are coupled to the system particle through a constant g. The Hahn echo experiments consists in preparing the system spin in the initial state $|y_s\rangle = (|\uparrow\rangle + i|\downarrow\rangle)/\sqrt{2}$, and then allowing free evolution for time τ. A π-pulse described by the Pauli operator σ^x is then applied to the system spin, and after free evolution for one more interval τ an echo is observed, which provides a direct measurement of single spin coherence. We would like to notice that the free evolution here means no additional driving fields exist, the coupling between the system and the spin-chain is always there.

We now follow the calculation [11] to derive an exact expression for the Hahn echo decay due to the system-chain couplings in equation (17). The density matrix for the whole system which will be used to calculate the Hahn echo is given by

$$\rho(\tau) = U(\tau)\rho_0 U^\dagger(\tau), \tag{19}$$

where $U(\tau)$ denotes the evolution operator [11]

$$U(\tau) = u(\tau)\sigma^x u(\tau), \quad u(\tau) = e^{-iH\tau}, \tag{20}$$

and ρ_0 is taken to be the initial state of the whole system

$$\rho_0 = |y_s\rangle\langle y_s| \otimes \rho_c(0) \tag{21}$$

with $\rho_c(0)$ denoting the initial state for the spin-chain. The Hahn spin echo envelope is then given by

$$v_E(\tau) = 2|\mathrm{Tr}\{(s^x + is^y)\rho(\tau)\}|. \quad (22)$$

In order to get an explicit expression for the Hahn echo envelope equation (22), we first write $u(\tau)$ in basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ (the eigenstates of σ^z), by noting that $[H_s, H_i] = 0$. This leads to

$$u(\tau) = \sum_{j=\uparrow,\downarrow} u_j(\tau)|j\rangle\langle j|, \quad (23)$$

with $u_j(\tau)$ satisfying,

$$i\hbar\frac{\partial}{\partial t}u_j(t) = H_j u_j(t),$$
$$H_j = -2\sum_{l=1}^{N}((1+\gamma)s_l^x s_{l+1}^x + (1-\gamma)s_l^y s_{l+1}^y + \lambda_j s_l^z), \quad (24)$$

where $\lambda_j = \lambda \pm g$, + and − correspond to $\downarrow$ and $\uparrow$, respectively. The free energy μ of the system which contributes only energy shifts to H_j would not affect the Hahn echo and has been omitted hereafter. For the system initially in state $|j\rangle$ ($j = \uparrow, \downarrow$), the dynamics and statistical properties of the spin-chain would be govern by H_j, it takes the same form as H_c but with perturbed field strengths λ_j. This perturbation to the spin-chain regardless of how small it is can be reflected in the Loschmidt spin echo decay [13], in particular at critical points. What behind the decay is the orthogonalization between two ground states obtained for two different values of external parameters [16]. The Hamiltonian H_j can be diagonalized by a standard procedure to be

$$H_j = \sum_k \omega_{j,k}\left(\eta_{j,k}^\dagger \eta_{j,k} - \frac{1}{2}\right), \quad (25)$$

which can be summarized in the following three steps. (1) The Wigner-Jordan transformation, which converts the spin operators into fermionic operators via the relation $a_l = (\prod_{m<l}\sigma_m^z)(\sigma_l^x + i\sigma_l^y)/2$, where σ_l is the Pauli matrix of the spin at site l; (2) the Fourier transformation, $d_k = \frac{1}{\sqrt{N}}\sum_l a_l \exp(-i2\pi lk/N)$; and (3) the Bogoliubov transformation, which defines the fermionic operators,

$$\eta_{j,k} = d_k \cos\frac{\theta_{j,k}}{2} - id_{-k}^\dagger \sin\frac{\theta_{j,k}}{2}, \quad (26)$$

where the mixing angle $\theta_{j,k}$ was defined by $\cos\theta_{j,k} = \varepsilon_{j,k}/\omega_{j,k}$, with $\omega_{j,k} = \sqrt{\varepsilon_{j,k}^2 + \gamma^2\sin^2\frac{2\pi k}{N}}$, and $\varepsilon_{j,k} = (\cos\frac{2\pi k}{N} - \lambda_j)$, $k = -N/2, -N/2+1, ..., N/2-1$. In this paper, we focus our attention on the periodic boundary condition, and the boundary term was ignored [17]. It is easy to show that $[\eta_{i,k}, \eta_{j,k}] \neq 0$ when $j \neq i$, i.e., the modes $\eta_{i,k}$ and $\eta_{j,k}$ do not commute (this is not the case for some special parameters discussed later on). This would result in the Hahn echo decay as you will see. With these results, the evolution operator $U(\tau)$ can be reduced to

$$U(\tau) = u_\uparrow(\tau)u_\downarrow(\tau)|\uparrow\rangle\langle\downarrow| + u_\downarrow(\tau)u_\uparrow(\tau)|\downarrow\rangle\langle\uparrow|, \quad (27)$$

where $u_j(\tau) = e^{-i\sum_k \omega_{j,k}(\eta_{j,k}^\dagger\eta_{j,k} - \frac{1}{2})\tau} \equiv \prod_k u_{j,k}(\tau)$, $j = \uparrow, \downarrow$, and $u_{j,k}(\tau) = e^{-i\omega_{j,k}(\eta_{j,k}^\dagger\eta_{j,k} - \frac{1}{2})\tau}$. After a simple algebra, we arrive at

$$v_E(\tau) = |\mathrm{Tr}_c(u_\downarrow^\dagger u_\uparrow^\dagger u_\downarrow u_\uparrow \rho_c(0))|, \quad (28)$$

where the trace is taken over the spin-chain. equation (28) can be simplified by noting that ($n_{j,k} = \eta_{j,k}^\dagger\eta_{j,k}$)

$$[n_{\uparrow,k}, n_{\downarrow,k}] = \frac{i}{2}\sin(\theta_{\uparrow,k} - \theta_{\downarrow,k})(\eta_{-k}\eta_k - \eta_k^\dagger\eta_{-k}^\dagger), \quad (29)$$

and consequently,

$$u_{\uparrow,k}^\dagger u_{\downarrow,k} = u_{\downarrow,k}u_{\uparrow,k}^\dagger + \hat{X}_k, \quad (30)$$

where $\hat{X}_k = (1 - e^{i\omega_{\uparrow,k}t})(1 - e^{-i\omega_{\downarrow,k}t})[n_{\uparrow,k}, n_{\downarrow,k}]$. Here $\eta_k = d_k\cos\frac{\theta_k}{2} - id_{-k}^\dagger\sin\frac{\theta_k}{2}$, and $\theta_k = \theta_{j,k}|_{\lambda_j=\lambda}$. Substituting equation (30) into $v_E(\tau)$, we get

$$v_E(\tau) = \prod_k |1 + \mathrm{Tr}_c[u_{\downarrow,k}^\dagger \hat{X}_k u_{\uparrow,k}\rho_c(0)]|. \quad (31)$$

The explicit expression for equation (31) can be obtained by choosing a specific initial state of the chain. We shall consider two initial states in this paper, (1) $\rho_c(0)$ is taken to be the fermionic vacuum state of H_c, (2) $\rho_c(0)$ is chosen to be a thermal state for the spin-chain. The fermionic vacuum state is exactly the ground state of H_c except $\gamma = 0$ [12], so the Hahn echo can signal the ground state properties in the many-body system. The fermionic vacuum state of H_c follows by the same steps summarized above. It is defined as the state to be annihilated by each operator η_k, namely $\eta_k|g(\gamma,\lambda)\rangle = 0$. After a few manipulations we obtain the Hahn echo envelope at zero temperature,

$$\begin{aligned} v_E(t) = \\ \prod_k \Big|1 &+ \frac{1}{4}\sin(\theta_{\uparrow,k} - \theta_{\downarrow,k})\sin(\theta_k - \theta_{\uparrow,k})|1 - e^{i\omega_{\uparrow,k}t}|^2 \\ &\times \left(1 - e^{-i\omega_{\downarrow,k}t} - \sin^2\frac{\theta_k - \theta_{\downarrow,k}}{2}|1 - e^{i\omega_{\downarrow,k}t}|^2\right) \\ &- \frac{1}{4}\sin(\theta_{\uparrow,k} - \theta_{\downarrow,k})\sin(\theta_k - \theta_{\downarrow,k})|1 - e^{i\omega_{\downarrow,k}t}|^2 \\ &\times \left(1 - e^{i\omega_{\uparrow,k}t} - \sin^2\frac{\theta_k - \theta_{\uparrow,k}}{2}|1 - e^{i\omega_{\uparrow,k}t}|^2\right)\Big|. \quad (32) \end{aligned}$$

With the above expressions, we now turn to study the Hahn echo at zero temperature. Since the XY model is exactly solvable and still present a rich structure, it offers a benchmark to test the properties of Hahn echo in the proximity of a quantum phase transition. For the XY model one can identify the critical points by finding the regions where the energy gap ω_k vanishes. Indeed, there are two regions in the λ, γ space that are critical. Namely, $\gamma = 0$ for $-1 < \lambda < 1$, and $\lambda = \pm 1$ for all γ. We first focus on the criticality in the XX model. The XX model that corresponds to $\gamma = 0$ has a criticality regime along

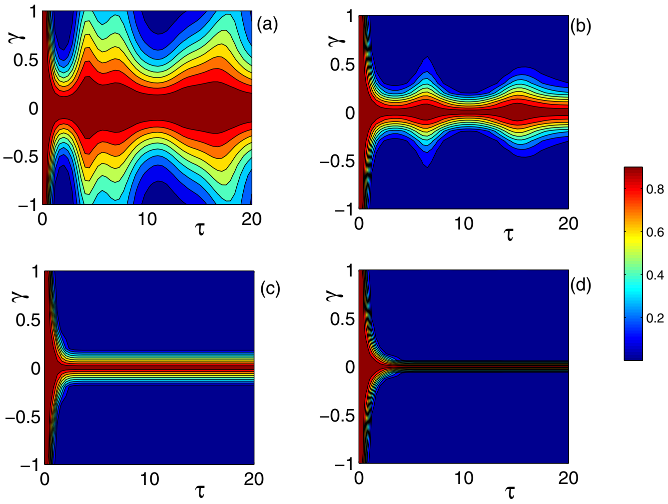

Fig. 1. (Color online) Hahn echo of the spin-1/2 particle *vs.* time τ and the anisotropy parameter γ. The spin-1/2 particle was coupled to a spin-chain described by the XY model. The parameters chosen are $N = 246$ sites, $g = 0.3$ and (a) $\lambda = 2$, (b) $\lambda = 1.5$, (c) $\lambda = 1$, and (d) $\lambda = 0.5$.

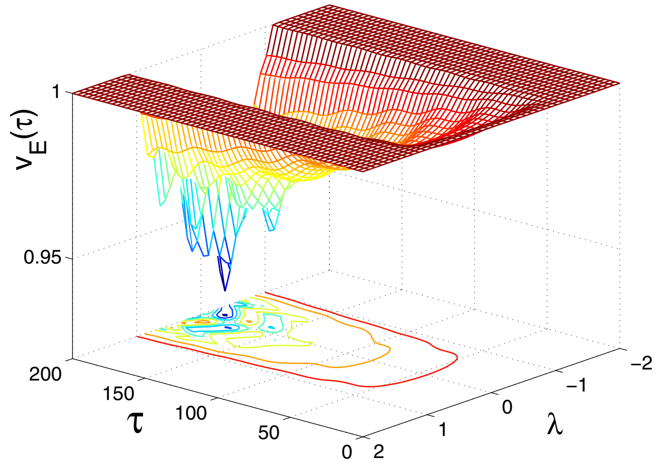

Fig. 2. (Color online) Hahn echo as a function of time τ and λ. The figure was plotted for $N = 246$ sites, $g = 0.1$ and $\gamma = 0.001$.

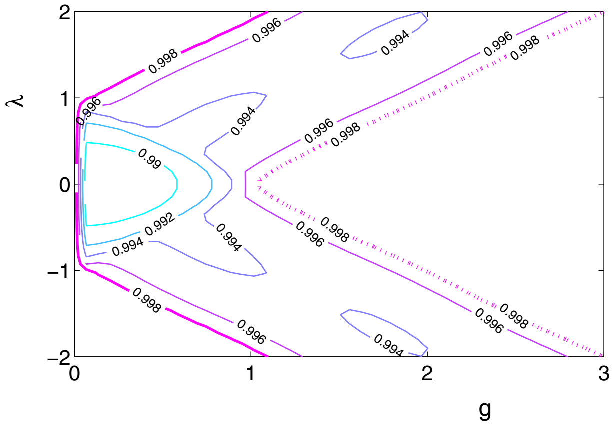

Fig. 3. (Color online) This figure was plotted to show the dependence of the critical points on the system-chain coupling constant g. Time $\tau = 50$, $N = 246$ sites, and $\gamma = 0.001$ were chosen for this figure.

the lines between $\lambda = 1$ and $\lambda = -1$ [18]. The critical points can be read out from the Hahn echo as shown in Figures 1 and 2. Figure 1 shows the Hahn echo as a function of time τ and the anisotropy parameter γ. Clearly, the Hahn echo takes a sharp change in the limit $\gamma \to 0$, this results can be understood by considering the value of $\theta_{j,k}$ and θ_k, which take 0 or π depending on the sign of $\cos(2\pi k/N) - \lambda_j$ and $\cos(2\pi k/N) - \lambda$, respectively. In either case, $\sin(\theta_k - \theta_{j,k}) = \sin(\theta_{\uparrow,k} - \theta_{\downarrow,k}) = 0$, this leads to $v_E(\tau) = 1$. Physically, when $\gamma = 0$, the particle number operators $n_{\uparrow,k}$ and $n_{\downarrow,k}$ commute, which implies that the perturbation from the system to the spin-chain does not excite the spin-chain, then the Hahn echo which characterizes the dephasing of the system remains unit. Figure 2 shows the Hahn echo $v_E(\tau)$ in the vicinity of critical points $\gamma \longrightarrow 0$ and $\lambda = \pm 1$. A sharp change among the line of $\lambda = \pm 1$ appears clearly.

We would like to notice that the Hahn echo $v_E(\tau)$ at critical points of $\gamma = 0$ and $\lambda = \pm 1$ does not depend on the chain-system coupling constant g, but in the vicinity of $\gamma = 0$, it does. This was shown in Figure 3, where we plotted the Hahn echo as a function of λ and g with $\gamma = 0.001$ (close to zero). As expected, the critical points have been shifted linearly by the coupling constant g. The white area in Figure 3 corresponds to $v_E(\tau) = 1$. In the region of $g > 2$ and $-1 < \lambda < 1$, $v_E(\tau)$ always equal to 1. This can be understood by examining the the definition of $\theta_{j,k}$ and θ_k. In this region, $\lambda_j = \lambda \pm g \geq 1$, leading to $\theta_{j,k} = \theta_k$ for any k in the limit $\gamma \to 0$. This results in $v_E(\tau) = 1$, which is a direct followup of equation (32).

Now we turn to study the criticality in the transverse Ising model ($\gamma = 1$ in the XY model). The ground state structure of this model change dramatically as the parameter γ is varied. We first summarize the ground states of this model in the limits of $|\lambda| \to \infty$, $|\lambda| = 1$ and $\lambda = 0$. The ground state of the spin-chain approaches a product of spins pointing the positive/negative z direction in the $|\lambda| \to \infty$ limit, whereas the ground state in the limit $\lambda = 0$ is doubly degenerate under the global spin flip by $\prod_{l=1}^{N} \sigma_l^z$. At $|\lambda| = 1$, a fundamental transition in the ground state occurs, the symmetry under the global spin flip breaks at this point and the chain develops a nonzero magnetization $\langle \sigma^x \rangle \neq 0$ which increases with λ growing. The above mentioned properties of the ground state are reflected in the Hahn echo as shown in Figure 4. In the limit $|\lambda| \to \infty$, $\theta_{j,k} = \theta_k = \pi/(-\pi)$, this results in $v_E(\tau) = 1$. In fact as Figure 4a shows, when $|\lambda| \geq 4$, $v_E(\tau)$ approaches 1 very well. With $|\lambda| \to 1$, the Hahn echo $v_E(\tau)$ tends to zero, this can be interpreted as the sensitivity of the spin-chain ground state to perturbations from the system-chain coupling at these points. The Hahn echo is a oscillating function of time τ around $\lambda = 0$. Due to the coupling to the spin-chain, the oscillation is damping, and eventually $v_E(\tau)$ tends to zero in the $\tau \to \infty$ limit. The difference between cases of $\gamma = 0$ and $\gamma = 1$ is that $[n_{\uparrow,k}, n_{\downarrow,k}] = 0$ for $\gamma = 0$, but it does not hold for $\gamma = 1$. This is the reason

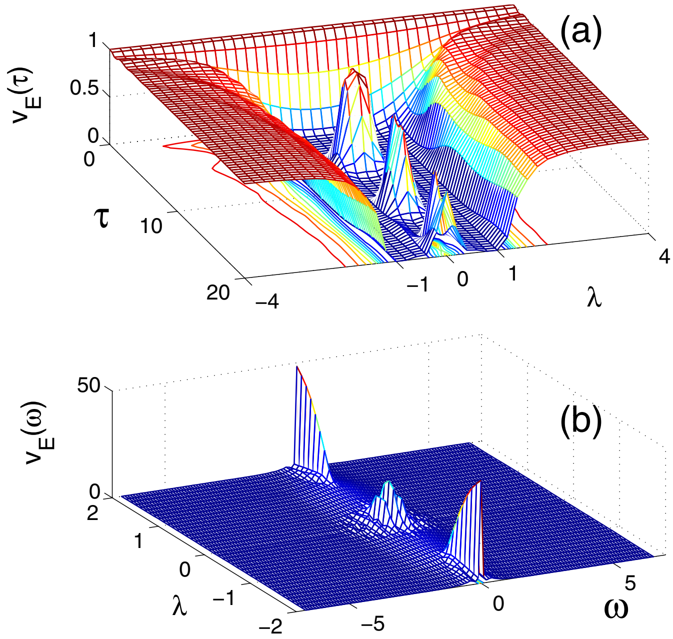

Fig. 4. (Color online) (a) Hahn echo *versus* time τ and λ with $\gamma = 1$. The other parameters chose are $g = 0.1$, $N = 246$ sites. (b) Discrete Fourier transformation of $v_E(\tau)$, with the same parameters as in (a).

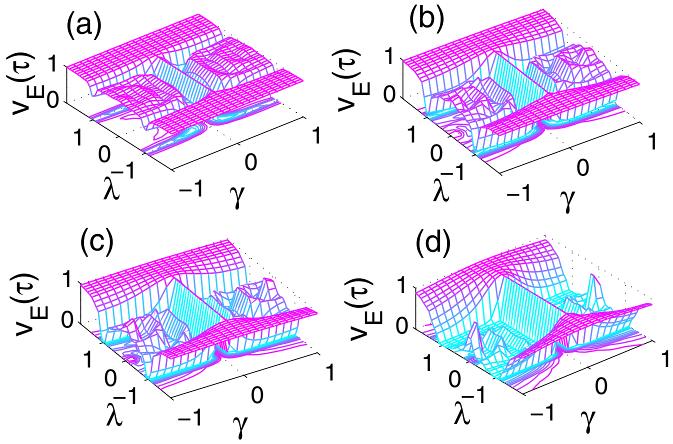

Fig. 5. (Color online) Hahn echo at time $\tau = 20$ with (a) $N = 250$, (b) $N = 1250$, (c) $N = 2500$, (d) $N = 10000$ sites, and $g = 0.01$.

why the Hahn echo takes different values at these critical points. Figure 4b is a discrete Fourier transformation of $v_E(\tau)$ with the same parameters as in Figure 4a. It would provides us the Hahn echo in the frequency domain. The ground state of the XY model is really complicated with many different regime of behavior [19], these are reflected in sharp changes in the Hahn echo across the line $|\lambda| = 1$ regardless of γ (as shown in Fig. 5), indicating the change in the ground state of the spin-chain from paramagnetic phase to the others.

The above connection between the Hahn echo and the criticality exists in a more general model with spin-system interactions

$$H_i = 4\sum_{l=1}^{N} s^z(g_l^z s_l^z + g_l^x s_l^x + g_l^y s_l^y), \tag{33}$$

where g_l^x, g_l^y, and g_l^z are coupling constants. The spin-system interaction equation (33) is general under the restriction $[H_i, H_s] = 0$ that we made in this example. The evolution operator $U(\tau)$ still takes the form of equations (20) and (23), but with

$$H_{\pm} = -2\sum_{l=1}^{N}[(1+\gamma)s_l^x s_{l+1}^x + (1-\gamma)s_l^y s_{l+1}^y + \lambda s_l^z \pm (g_l^z s_l^z + g_l^x s_l^x + g_l^y s_l^y)], \tag{34}$$

instead of equation (23). It is easy to show that the Hahn echo can be written as

$$v_E(\tau) = |\langle G_c|e^{-iH_-\tau}e^{iH_+\tau}|G_c\rangle|^2. \tag{35}$$

For small coupling constants g_l^x, g_l^y and g_l^z, it has been proved that the overlapping $|\langle G_c|e^{-iH_-\tau}e^{iH_+\tau}|G_c\rangle|$ would well signal the critical points in the one-dimensional XY model, leading to the claim that the Hahn echo $v_E(\tau)$ can be used as a marker for criticality. We discuss now the finite-N effect on the Hahn echo in this model. For this purpose, we reexamine the Hahn echo $v_E(\tau)$ given by equation (32), which is a product of N terms. Noticing that each term is positive and smaller than 1, we claim that with N increasing, the changes in the vicinity of the critical points become sharper. To confirm this, we plot $v_E(\tau)$ versus γ and λ with a fixed $\tau = 20$ and several N in Figure 5. Clearly, the larger N the sharper the change in the vicinity of $\gamma = 0$ and $|\lambda| = 1$. This conclusion holds for different τ.

Next, we consider the second example where the spin-quantum system coupling H_i does not commute with the free Hamiltonian of the spin-1/2 particle, indicating that the energy of the spin-1/2 particle is not conserved. We will calculate the Hahn echo according to the general formalism presented in Section 2. The one-dimensional XY spin chain is still chosen as the quantum system, while the spin-system coupling H_i takes the same form in equation (8). To simplify the calculation, we chose $|\psi_c(0)\rangle = c_0|E_0\rangle + c_1|E_1\rangle$ ($|c_0|^2 + |c_1|^2 = 1$) as the initial state of the many-body system, where $|E_0\rangle$ and $|E_1\rangle$ denote the ground state and first excited state of the free many-body system $H_c(\lambda)$, respectively. The spin-1/2 particle initially is prepared in the state $|y_s\rangle = 1/\sqrt{2}(|\uparrow\rangle + i|\downarrow\rangle)$, as we did in the first example. The Hahn echo can then be expressed as

$$v_E(\tau) = 2|\mathrm{Tr}(\sigma^+\rho_s(\tau))|, \tag{36}$$

where

$$\rho_s(\tau) = u(\tau)\sigma^x u(\tau)|y_s\rangle\langle y_s|u^\dagger(\tau)\sigma^x u^\dagger(\tau),$$

$$u(\tau)(...)u^\dagger(\tau) = |c_1|^2 U_1(\tau)(...)U_1^\dagger(\tau) + |c_0|^2 U_0(\tau)(...)U_0^\dagger(\tau),$$

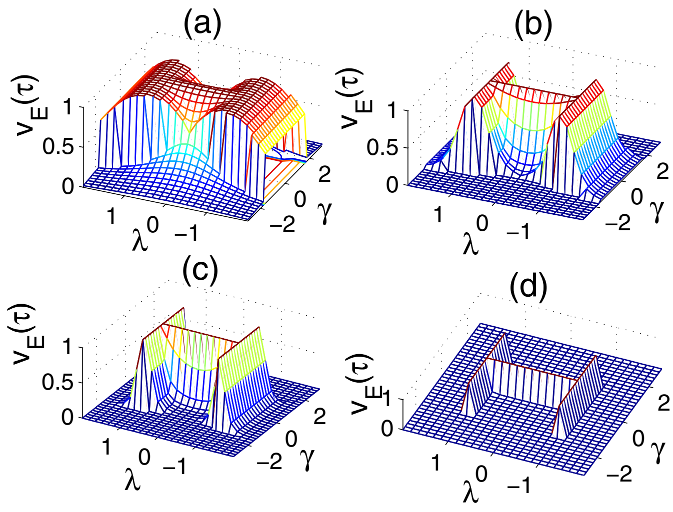

Fig. 6. (Color online) Hahn echo at time (a) $\tau = 30$, (b) $\tau = 100$, (c) $\tau = 200$ and (d) $\tau = 1000$ with $N = 246$ sites, and $g_x = g_y = g_z = 0.1$ and $c_0 = c_1 = \frac{1}{\sqrt{2}}$.

and $U_j(\tau) = e^{-iH_j\tau}, j = 0, 1$. H_j was given in equation (10). We have performed extensive numerical simulations, selective results are presented in Figure 6. Figure 6a was plotted for the Hahn spin-echo $v_E(\tau)$ as a function of λ and γ at time $\tau = 30$. It shows that $v_E(\tau) = 1$ at $\gamma = 0$ or $|\lambda| = 1$, otherwise $v_E(\tau) < 1$. Figures 6b–6d are for $\tau = 100, 200$, and 1000, respectively. Clearly, $v_E(\tau) = 1$ with $\gamma = 0$ or $|\lambda| = 1$. These results tell us that the Hahn echo can well signal the critical points $\gamma = 0$ and $|\lambda| = 1$. The longer the time τ, the sharper the change of $v_E(\tau)$ in the vicinity of the critical points. The same finite-N effect as in the first example can be found in our numerical simulations for this model.

4 Remarks on low temperature effects and conclusion

Up to now, we did not consider the temperature effect. Finite temperature is the regime to which all experiments being confined, but what is the finite temperature effect on the Hahn echo? In the following, we shall consider this problem by studying the contributions of one- and two-particle excitations to the Hahn echo. The coupling of the spin-1/2 particle to the spin-chain is of pure dephasing. Taking a thermal state $\rho_c^T(0) = \frac{1}{z}e^{-\beta H_c}$ ($\beta = \frac{1}{k_B T}$) as the initial state of the spin-chain, the Hahn echo envelope can be written as

$$v_E^T(\tau) = \prod_k |1 + \frac{1}{z}\sum_n e^{-\beta\Omega_n}\langle n|u_{\downarrow,k}^\dagger \hat{X}_k u_{\uparrow,k}|n\rangle|, \qquad (37)$$

where $|n\rangle$ and Ω_n denote the eigenstate and corresponding eigenvalue of H_c, respectively. z is the partition function. We shall restrict our consideration to the contribution from one- and two-particle excitations of the chain,

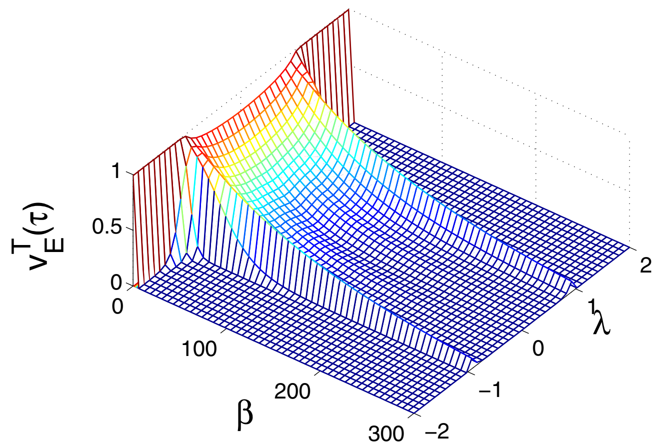

Fig. 7. (Color online) Contribution of the two-particle excitation to Hahn echo at time $\tau = 40$. The other parameters chosen are $N = 246$ sites, $\gamma = 0.01$, and $g = 0.02$.

namely,

$$|n\rangle \in \{\eta_{j,k}^\dagger|g(\gamma,\lambda)\rangle, \eta_{i,k_1}^\dagger\eta_{j,k_2}^\dagger|g(\gamma,\lambda)\rangle\}, \quad \text{for } k_1 \neq k_2, \text{ or } i \neq j, \qquad (38)$$

with k, k_1 and k_2 ranging from $-N/2$ to $N/2-1$. It is not difficult to show that there are no contribution from the one particle excitation, because $\hat{X}_k$ creates or annihilates two particles with k and $-k$ jointly. The numerical results presented in Figure 7 show the contribution of the two-particle excitation to the Hahn echo, we find that the quantum critical points can influence the Hahn echo at a finite temperature. For the parameters chosen in Figure 7, the contribution form the thermal excitation is larger than that from quantum fluctuation when $\beta < 72 = \beta_c$. Here we have scaled out an overall energy scale denoted by J. J may be taken to be of order 1000 K, that is the order for the antiferromagnetic exchange constant of the Heisenberg model. It yields $T_c \sim 14$ K corresponding to parameters chosen in Figure 7. For the transverse Ising model $\gamma = 1$, β_c is of order 10, we obtain $T_c \sim 100$ K in this situation with the other parameters being the same as in Figure 7. Notice that the study here is based on the Hahn echo (a dynamical quantity), this would differ from the investigation based on thermodynamics [20]. We would like to notice that the discussion on the finite temperature effect was limited to very low temperatures, because only one- and two-quasiparticle excitations were included. Nevertheless, it is interesting because it also sheds light on the contribution to Hahn echo from the first excited states, which have the same energy as the ground state $\rho_c(0)$ of H_c at critical points. The results presented in Figure 7 show that those contributions tend to zero with $T \to 0$.

In conclusion, by discussing the Hahn spin echo in the spin-1/2 particle coupled to the many-body systems, the relation between the Hahn echo and the critical points was established. The relation not only provides an efficient theoretical tool to study quantum phase transitions, but also proposes a method to measure the critical points in experiments. Up to two-particle excitations, we have also studied the influence of thermal fluctuation on the Hahn echo, it would shed light on the low temperature (with respect to the overall energy scale J) effects on the Hahn echo. The limitation of this discussion is that the coupling between the spin and the quantum system is assumed weak, and as we have shown, the Hahn echo could not reflect the critical points of the quantum system that is initially in its ground state with spin-system coupling satisfying $[H_c, H_i] = 0$.

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